Midpoint of Symmedian chord

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In this paper, we discuss the properties of a unique and pretty special point on the symmedian, which happens to be the midpoint of the respective symmedian chord; and some rich configurations associated with it. It is popularly known as the "Dumpty point" in the community; therefore, we will also call it the same, throughout. We will further explore some configurations associated with it, and look into some related examples and problems.

Contents

Acknowledgements

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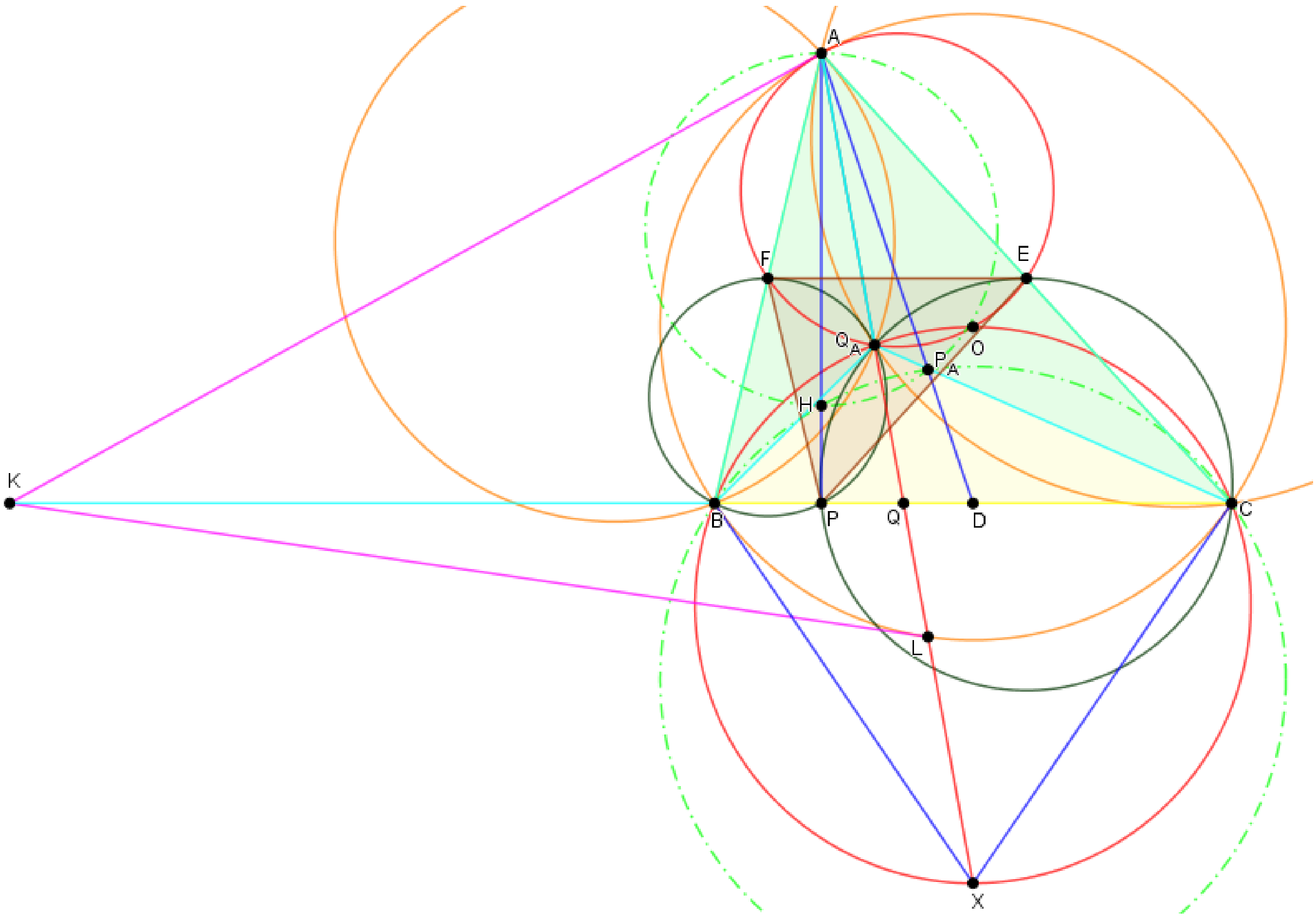
Prerequisites

Readers are expected to be familiar with conventional notations, properties of Symmedians, HM point, Complete Quadrilaterals, and basic Projective geometry.

1 Introduction and Characterizations

Notations:

- Let D, E, F be the midpoints of BC, CA, AB, respectively, and P to be the foot of A-altitude on BC.
- Let H, O, G denote the orthocenter, circumcenter, centroid of $\triangle ABC$, respectively, and H' be the reflection of H in BC.
- The tangents at B and C to (ABC) meet at X, and hence, AX constitutes the A-symmedian in $\triangle ABC$.
- $AX \cap (ABC) = L$, and $AX \cap BC = Q$.
- K is the intersection of tangents at A, L to (ABC).
- Let $A' \in (ABC) \neq A$, such that $AA' \parallel BC$.
- P_A is the A-HM point in $\triangle ABC$, that is, the foot from H to AD.



Other notations will be shown accordingly in latter diagrams.

Definition
We define midpoint of symmedian chord, Q_A, as the A-Dumpty point.

This very definition yields our following characterization.

Characterization 1. In triangle ABC, point Q_A satisfies following angle relations:

$$\angle Q_ABA = \angle Q_AAC \text{ and } \angle Q_ACA = \angle Q_AAB.$$

Proof. From the **harmonic** condition of $(ABLC)$ we get that BC concurs with the tangents at A and L at some point K. So, we have BQ as a symmedian in $\triangle ABL$, which gives

$$\angle ABQ_A = \angle LBQ = \angle LAC = \angle Q_AAC.$$

Similarly, CQ being symmedian in $\triangle ACL$ gives

$$\angle ACQ_A = \angle LCB = \angle LAB = \angle Q_AAB. \qquad \square$$

Note that, the above also gives $\triangle BQ_AA \sim \triangle BLC \sim \triangle AQ_AC$. (Also here, apart from the Harmonic way, one can also note that, as AL is the polar of K wrt[1] (ABC), and X lies on AL, hence, from *La-Hire's theorem*, K lies on the polar of X wrt (ABC), which is BC.)

Characterization 2. There exists a spiral similarity at Q_A that sends BA to AC.

Proof. As we have $\triangle AQ_AB \sim \triangle CQ_AA$, so, we simply get the common vertex Q_A as the unique center of spiral similarity. $\square$

We also get

$$\frac{AQ_A}{CQ_A} = \frac{Q_AB}{Q_AA} \implies AQ_A^2 = CQ_A \cdot BQ_A. \tag{1}$$

This proves to be really helpful in quite some places. Now, look into the below one.

Lemma 1
$\triangle BLQ_A \sim \triangle BCA \sim \triangle LCQ_A$.

Characterization 3. Q_A is the isogonal conjugate of the P_A.

Proof.

$$\angle Q_ABA = \angle Q_AAC = \angle BAP_A = \angle CBP_A$$

where the last equality follows from properties of A-HM point, and the one before it as $\angle BAQ_A = \angle CAP_A$. $\square$

[1]stands for "with respect to", and the abbreviation will be used henceforth

Characterization 4. In $\triangle ABC$ with circumcenter O, the circle with diameter $\overline{AO}$ and (BOC) intersect again on the A-symmedian at a point Q_A.

Proof. We observe that

$$\angle Q_A AB + \angle Q_A AC = \angle A \implies \angle Q_A AB + \angle Q_A BA = \angle A \implies \angle AQ_A B = 180^\circ - \angle A.$$

Similarly, we get $\angle AQ_A C = 180^\circ - \angle A$. Hence,

$$\angle BQ_A C = 2\angle A = \angle BOC.$$

So, $Q_A \in (BOC)$. (It goes without saying, as $X \in (BOC)$, so, **Charac 4** would've hold fine for (BXC) as well; and thus $Q_A \in (BXC)$. Also, from there, we could have easily got $Q_A \in (BXCO)$ as $OQ_A \perp AQ_A$, but the stated is just another way.)

Lastly, for the remaining part, note that $OQ_A \perp AQ_A$, and as $OE \perp CA, OF \perp AB$, it's evident that $Q_A \in (AEF)$. □

There is another way to prove this 90° fact, but it uses $\sqrt{bc}$ inversion, which we will explore in later sections.

Characterization 5. Let ω_B denote the circle through B tangent to AC at A, and ω_C be the circle through C tangent to AB at A. Then, ω_B and ω_C intersect again at Q_A.

Proof. Using the angle conditions, we note that $(BQ_A A)$ is tangent to AC at A, appealing to the Alternate Segment theorem. Likewise, we get $(CQ_A A)$ tangent to AB at A; which is what we desired. □

(Converse of **Charac 5**). Assuming the $Q_A = \omega_B \cap \omega_C \neq A$, prove that Q_A is the midpoint of AL. (Use Equation (1) and **Lemma 1**.)

Characterization 6. (BFP) and (CEP) intersect again at Q_A.

Proof.

Claim — Quadrilateral $BFQ_A P$ is cyclic.

Proof.

$$\begin{aligned}\angle FQ_A B &= 360^\circ - \angle BQ_A O - \angle OQ_A F \\ &= (180^\circ - \angle BQ_A O) + (180^\circ - \angle OQ_A F) \\ &= \angle OCB + \angle OAF \\ &= \angle B = \angle FPB.\end{aligned}$$

where the last step follows as F is the midpoint of $\overline{AB}$ in right $\triangle APB$. □

Analogously, we get $CEQ_A P$ is cyclic, and thus, Q_A as the second intersection of the two circles. □

Lastly, observe that, Q_A is the P-HM point in $\triangle PEF$.

Note that we can get any Characterization from any other and so on, using Phantom points, etc. Now, the reader might want to take a look at the problems given below.

Problem 1 (AIME 2019 II/11). Triangle ABC has side lengths $AB = 7, BC = 8$, and $CA = 9$. Circle ω_1 passes through B and is tangent to line AC at A. Circle ω_2 passes through C and is tangent to line AB at A. Let K be the intersection of circles ω_1 and ω_2 not equal to A. Then $AK = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 2 (Polish Second Round 1999/4, All-Siberian Open School 2016-17/11.3). Inside acute triangle ABC, let P be a point, distinct from the circumcenter of the triangle, such that $\angle PAB = \angle PCA$ and $\angle PAC = \angle PBA$. Prove that $\angle APO$ is right.

Problem 3 (Dutch IMO TST 2013/1.3). Fix a triangle ABC. Let Γ_1 the circle through B, tangent to edge in A. Let Γ_2 the circle through C tangent to edge AB in A. The second intersection of Γ_1 and Γ_2 is denoted by D. The line AD has second intersection E with the circumcircle of $\triangle\ ABC$. Show that D is the midpoint of the segment AE.

Problem 4 (St Petersburg 1996, Moscow 2011/2 Oral Team IX). Inside triangle ABC, with $\angle A = 60^\circ$, a point T is chosen such that $\angle ATB = \angle ATC = 120^\circ$. Let M, N be the midpoints of sides AB, AC, respectively. Prove that the quadrilateral $AMTN$ is cyclic.

A variant of the above problem is as follows:

Problem 5 (All Russian Grade 9 2021/6). Given is a non-isosceles triangle ABC with $\angle ABC = 60^\circ$, and in its interior, a point T is selected such that $\angle ATC = \angle BTC = \angle BTA = 120$. Let M the intersection point of the medians in ABC. Let TM intersect (ATC) at K. Find TM/MK.

The reader might have guessed by now, that all the above ones are indeed trivialized by the content of the above section.

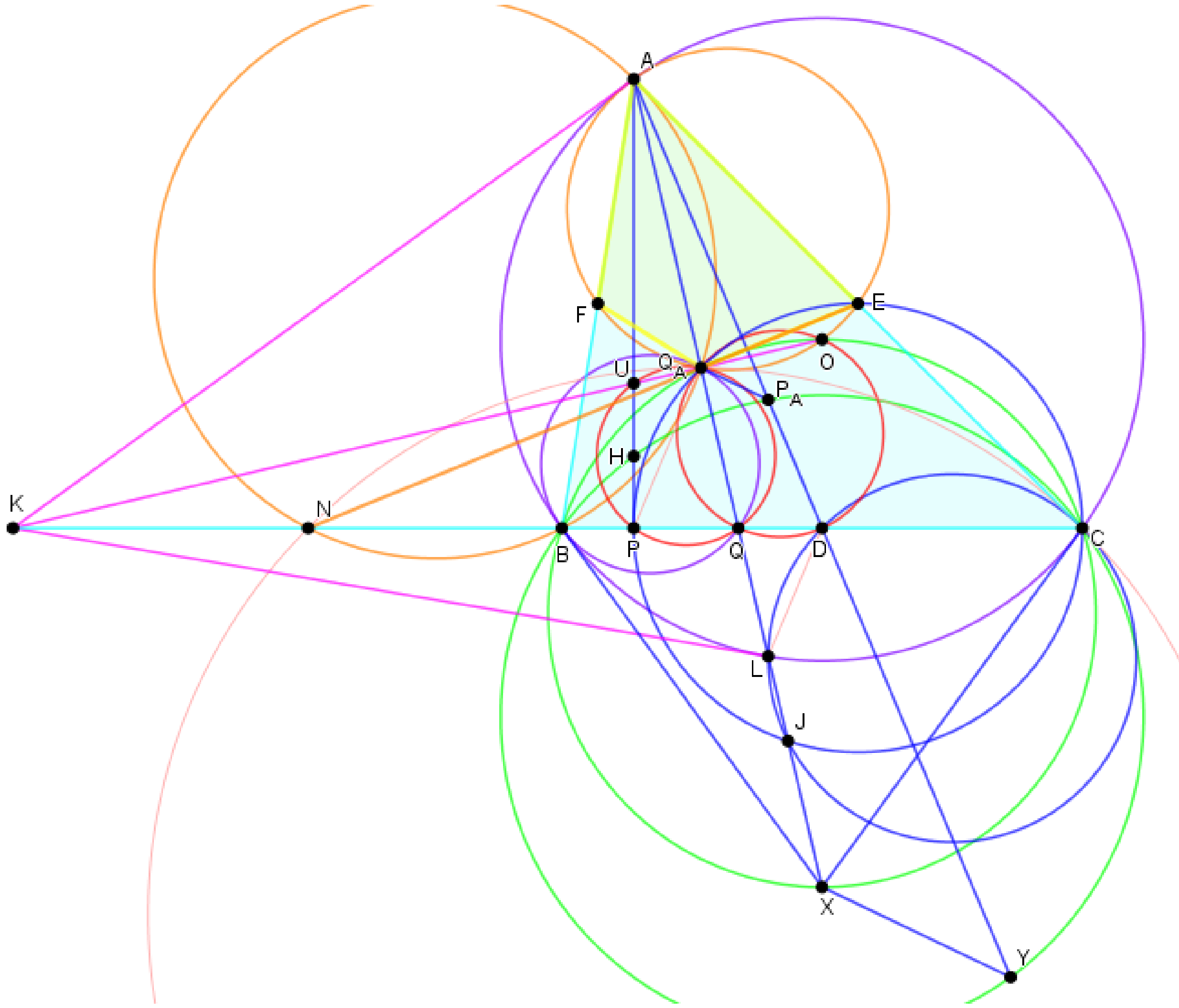
Problem 4 and 5 reminds of something?

Lemma 2

So, we observe that in a 60° vertex triangle, the Dumpty point wrt that particular vertex, coincides with the Fermat point of the triangle.

2 More Interesting Properties

Next, we will observe some collinearities, concyclicities, and so on and so forth.



Lemma 3
(BQQ_A) is internally tangent to (ABC).

Proof.

$$\angle BQ_AQ = \angle BAQ_A + \angle ABQ_A = \angle BAQ_A + \angle Q_AAC = \angle A = \angle XBC = \angle XBQ.$$

Therefore, $\overline{XB}$ is tangent to both (ABC) and (BQQ_A) at B. □

Lemma 4
O, Q_A, K are collinear.

Proof. Since K is the intersection of tangents at A, L to (ABC), this follows straight from symmetry. □

Without an Harmonic quadrilateral every picture is (almost) incomplete! Whether it's $\triangle BIC$ related configurations or "the foot of altitude in a contact triangle" related, so here goes our one.

Lemma 5
AFQ_AE is harmonic.

Proof. Note that (ABC) and (AFE) are tangent at A by simple homothety, which in turn takes the tangents at B, C to (ABC) to tangents at F, E to (AFE), respectively, and so on. With the intersection of the tangents still on the AX, and as we already have AFQ_AE cyclic, we're done. □

Either way just note that the same homothety maps the entire harmonic quadrilateral $(ABLC)$ to quadrilateral (AFQ_AE), yielding the latter to be harmonic as well.

Lemma 6
Quadrilateral Q_AOQD is cyclic, and so are the points Q_A, Q, P, U, where $U = OQ_A \cap AP$.

Proof. As both $\angle ODQ$ and $\angle OQ_AA$ are right, we're done with the first one. For the latter, we note that $\angle UQ_AQ = 180° - \angle OQ_AQ = 90° = \angle UPQ$. □

Suppose, AD meets (BHC) at Y.

Lemma 7
$Q_AP_A \parallel XY$.

Proof.

Claim — $ABYC$ is a parallelogram.

Proof.

$$\angle BCY = \angle BHY = 90° - \angle BYH = 90° - \angle BCH = \angle B. \quad (*)$$

Similarly, we get for the other part. □

Now,

$$\angle AP_AB = 180° - \angle BP_AY = 180° - \angle BCY \overset{(*)}{=} 180° - \angle B = \angle A + \angle C = \angle ACX.$$

And, as $\angle BAP_A = \angle XAC$, we get $\triangle BAP_A \sim \triangle XAC$. Analogously, $\triangle BAQ_A \sim \triangle YAC$. Hence,

$$\frac{BA}{AP_A} = \frac{XA}{AC}, \text{ and } \frac{BA}{AQ_A} = \frac{YA}{AC} \implies \frac{AP_A}{YA} = \frac{AQ_A}{XA}$$

whence, we get the required. □

(This same point Y is used in the proof of the very first property of P_A - that it's the second intersection of (AH) and (BHC) on AD. In that Y is defined as the point such that $ABYC$ forms a parallelogram, and later it's shown that $Y \in (BHC)$, and that it's the antipode of H in (BHC). Then it's used in angle chase to show that $P_A \in AD$.)

Let EQ_A intersect BC at N.

Lemma 8
N lies on ω_B.

Proof. Let ω_B intersect BC again at N'. We know that $\angle Q_ACA = \angle Q_AAB$, so

$$\angle Q_ACA = \angle Q_AN'C,$$

and also,

$$\angle Q_AAC = \angle Q_AN'A.$$

Hence, CA is tangent to both (AQ_AN') (which is ω_B), and $(N'Q_AC)$. Now, $N'Q_A$ being the radical axis bisects the common external segment, and thus passes through E, forcing $N' = N$. □

Lemma 9
The intersection (CEQ_AP) and (CDL) distinct from C lies on the A-symmedian (J in the diagram).

Proof. We begin with a claim.

Claim — $Q_AP \parallel DL$.

Proof.

$$\angle PQ_AL = 180^\circ - \angle FQ_AP - \angle AQ_AF = 180^\circ - (180^\circ - \angle B) - \angle AEF = \angle B - \angle C.$$

Now, note that as $\angle CLD = \angle BLQ = \angle BLA = \angle BCA$, so,

$$\angle DLQ_A = \angle CLQ - \angle CLD = \angle CBA - \angle BCA = \angle B - \angle C,$$

and thus, $Q_AP \parallel DL$. □

Now, let (CEQ_A) intersect A-symmedian at J. Then, we prove $CDLG$ is cyclic.

$$\angle CJL = \angle CJQ_A = \angle CPQ_A = \angle QPQ_A = \angle QDL.$$ □

(The above one might seem to be sudden, and yes indeed, no such motivation for it. I discovered it while exploiting the diagram, and in the hunt to extract any further property left behind in the diagram.)

Remark. It's quite obvious to note that many of the lemmas and problems hold with respect to configurations oriented at any vertex, whether it's B or C, wrt the A-Dumpty point, in $\triangle ABC$.

We now encourage the reader to try the problems given below.

Problem 6 (Morocco 2015). Let ABC be a triangle and O be its circumcenter. Let T be the intersection of the circle through A and C tangent to AB and the circumcircle of BOC. Let K be the intersection of the lines TO and BC. Prove that KA is tangent to the circumcircle of ABC.

Problem 7 (AIME 2019 I/15). Let $\overline{AB}$ be a chord of a circle ω, and let P be a point on the chord $\overline{AB}$. Circle ω_1 passes through A and P and is internally tangent to ω. Circle ω_2 passes through B and P and is internally tangent to ω. Circles ω_1 and ω_2 intersect at points P and Q. Line PQ intersects ω at X and Y. Assume that $AP = 5$, $PB = 3$, $XY = 11$, and $PQ^2 = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

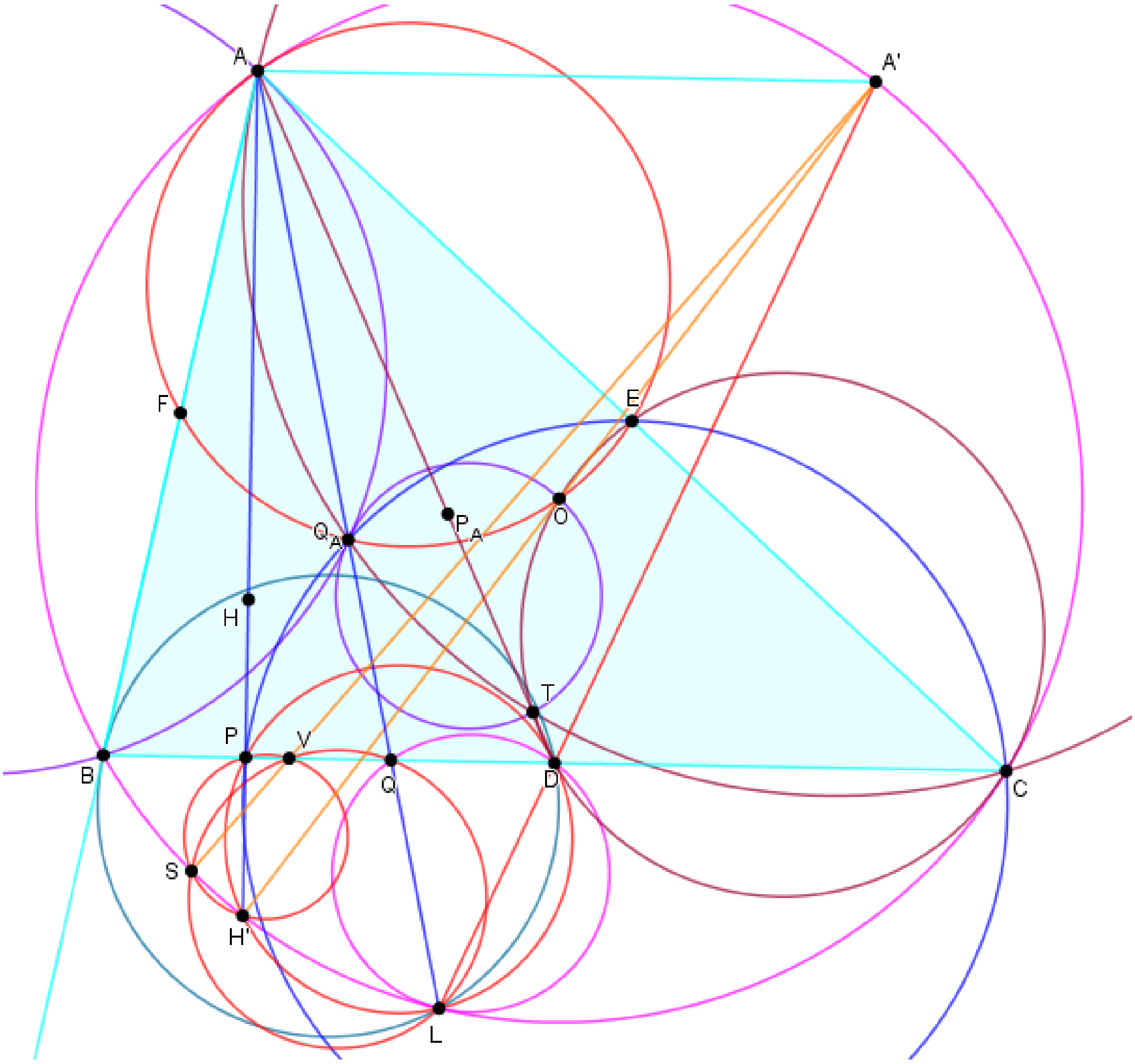
Problem 8 (BxMO 2020/3). Let ABC be a triangle. The circle ω_A through A is tangent to line BC at B. The circle ω_C through C is tangent to line AB at B. Let ω_A and ω_C meet again at D. Let M be the midpoint of line segment $[BC]$, and let E be the intersection of lines MD and AC. Show that E lies on ω_A.

Problem 9 (INAMO Shortlist 2015 G8). ABC is an acute triangle with $AB > AC$. Γ_B is a circle that passes through A, B and is tangent to AC on A. Define similar for Γ_C. Let D be the intersection Γ_B and Γ_C and M be the midpoint of BC. AM cuts Γ_C at E. Let O be the center of the circumscibed circle of the triangle ABC. Prove that the circumscibed circle of the triangle ODE is tangent to Γ_B.

Problem 10 (ELMO 2014/5, *Sammy Luo*). Let ABC be a triangle with circumcenter O and orthocenter H. Let ω_1 and ω_2 denote the circumcircles of triangles BOC and BHC, respectively. Suppose the circle with diameter $\overline{AO}$ intersects ω_1 again at M, and line AM intersects ω_1 again at X. Similarly, suppose the circle with diameter $\overline{AH}$ intersects ω_2 again at N, and line AN intersects ω_2 again at Y. Prove that lines MN and XY are parallel.

Problem 11 (ELMO Shortlist 2012 G7, *Alex Zhu*). Let $\triangle ABC$ be an acute triangle with circumcenter O such that $AB < AC$, let Q be the intersection of the external bisector of $\angle A$ with BC, and let P be a point in the interior of $\triangle ABC$ such that $\triangle BPA$ is similar to $\triangle APC$. Show that $\angle QPA + \angle OQB = 90°$.

Continuing with some related properties and results, which don't actually involve the Q_A point, but are nice to observe, here we have the below ones.



Lemma 10
L, D, A' are collinear.

Proof. Note that

$$\angle ALD = \angle BLD - \angle BLQ = \angle CLQ - \angle BLA = \angle B - \angle C$$

where $\angle BLD = \angle CLQ$, as LA constitutes the L-symmedian in $\triangle LBC$. But, $\angle ALA' = \angle ACA' = \angle B - \angle C$, so we're done. □

Lemma 11
(BDL) is tangent to AB at B.

Proof. $\angle ABD = \angle A'CB = \angle A'LB = \angle DLB$. □

Similarly, (CDL) is tangent to CA at C.

Lemma 12
(QDL) is internally tangent to (ABC), at L.

Proof. $\angle KLQ = \angle KLA = \angle LA'A = \angle LDQ$. □

Lemma 13
Points A, O, D, L, K lie on a circle with center as the midpoint OK. (See **Lemma 4**.)

Proof. $\angle OAK, \angle OLK, \angle ODK$ all are 90°. □

Lemma 14
P, H', L, D are concyclic.

Proof. $\angle H'LD = \angle H'LA' = 90^\circ = \angle APD$. □

Lemma 15
Let V be a point on BC, and $A'V$ intersect (ABC) again at S, then, P, V, H', S are concyclic.

Proof. $\angle APC = 90^\circ = \angle H'PC = \angle H'AA' = \angle HSA' = \angle HSV$. □

(I wouldn't have got the above one, hadn't I joined A' and P_A mistakenly, instead of A' and G while constructing the GP line configuration, which readers will found later. Later, it was found that there is nothing special about the point, V can be any point on BC, and that led to the statement presented.)

Lemma 16
S, V, Q, L are concyclic as well.

Proof. $\angle LQD = \angle LAA' = \angle LSA' = \angle LSV$. □

Lemma 17
$(CEOD), \omega_C$ and the A-median share a common point (T in the diagram).

Proof. Let AD intersect (CEO) at T. Then, we show T lies on ω_C.

Claim — (AFE) and (CED) are reflection of each other across OE; Q_A, T get swapped under this reflection.

Proof. It suffices to show $\angle OET = \angle OEQ_A$, as (AFE) and (CED) are reflections by symmetry. So, firstly we note that

$$\begin{aligned}\angle OAQ_A = \angle OAQ &= \angle A - \angle BAQ - \angle CAO \\ &= \angle A - \angle CAP_A - (90^\circ - \angle B) \\ &= 90^\circ - (\angle C + \angle CAD) \\ &= 90^\circ - \angle ADB = \angle ODA.\end{aligned}$$

Now,

$$\angle OET = \angle ODT = \angle ODA = \angle OAQ_A = \angle OEQ_A,$$

whence the desired. □

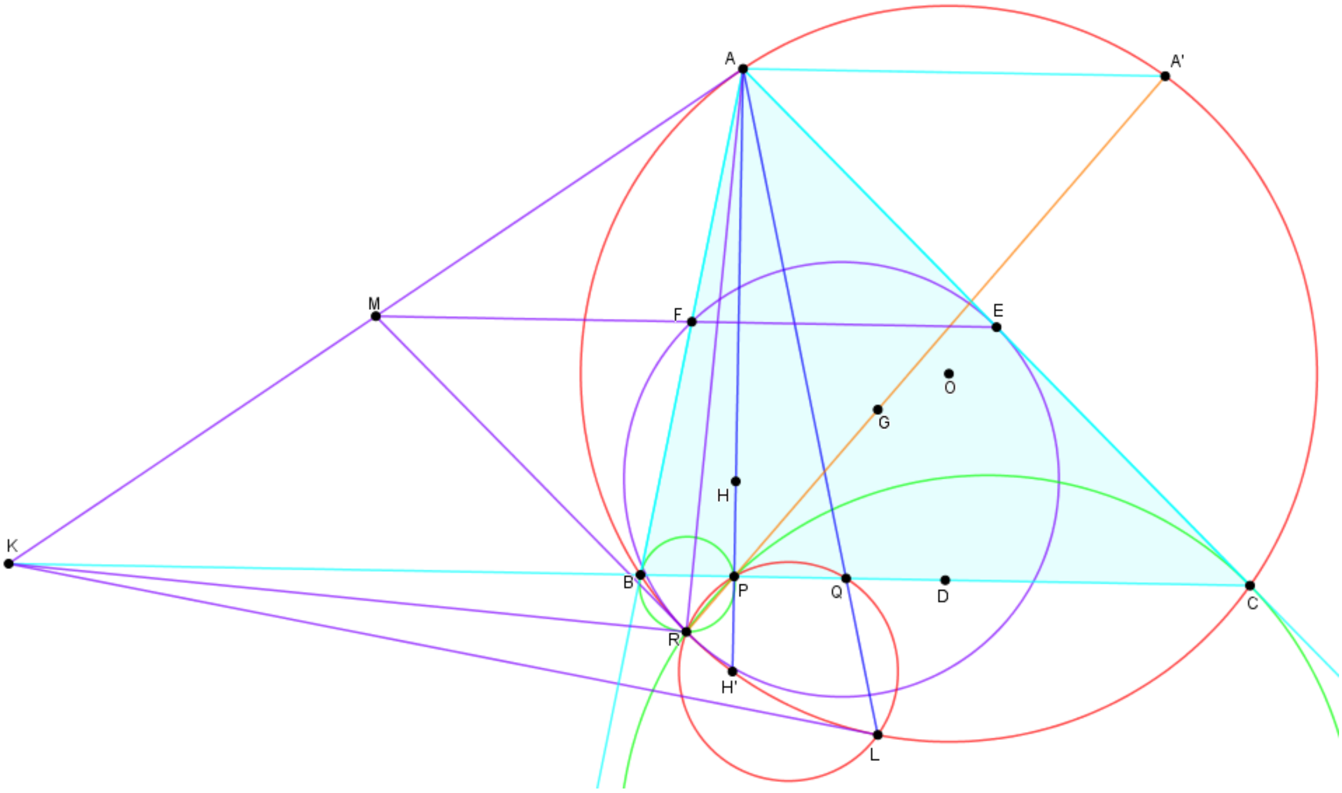
That yields Q_A, T are reflections across OE! Finally, to finish observe that

$$\angle BAT = \angle BAP_A = \angle CAQ_A = \angle EAQ_A = \angle ECT = \angle ACT. \qquad \square$$

Motivation for this point T came from **INAMO Shortlist 2015 G8**. Similar things hold with everything in respect to vertex B in $\triangle ABC$.

GP line

Readers are advised to enjoy & explore the configuration given below, like the above ones.



Lemma 18
Let ω be a circle through E and F that is tangent to (ABC) at a point $R \neq A$. Let us re-define K here, as the tangent to (ABC) at A which intersects line BC at K. Prove that

1. (AoPS). $AR \perp RK$, that is, AR is the polar of M wrt Ω., where M is the midpoint of AK.
2. (IMO Shortlist 2011 G4). G, P and R are collinear.
3. (USA TST 2018/2). $\angle BRE = \angle FRC$.
4. RF, RE are the R-symmedian of triangles BRP, CRP.
5. R, P, Q, L are concyclic.
6. (BPR) is tangent to AB at B, and likewise, (CPR) is tangent to CA at C. (Similar to **Lemma 11**.)

(For more configurations and details related to this point R, see here.)

3 Walk-through Some Contest Examples

As this work is dedicated towards Dumpty points, so it's indeed pointless to say "Take a guess! What can be the point?", etc.

Example 19 (Macedonia 2017/4)
Let O be the circumcenter of the acute triangle ABC ($AB < AC$). Let A_1 and P be the feet of the perpendicular lines drawn from A and O to BC, respectively. The lines BO and CO intersect AA_1 in D and E, respectively. Let F be the second intersection point of (ABD) and (ACE). Prove that the angle bisector of $\angle FAP$ passes through the incenter of $\triangle ABC$.

Walkthrough.

(a) Have a close look at the cyclic quadrilaterals of $(AFBD)$ and $(AEFC)$.

(b) Chase the angles surrounding D and E, to get $\triangle ABF \sim \triangle CAF$. In other words, get (ADB) tangent to AC at A, and similarly (AEC) tangent to AB at A.

(c) So, we get F as the A-Dumpty point in $\triangle ABC$, and thus AF and AP as isogonals.

The below one is a very popular and known example that comes whenever talking about Dumpty point.

Example 20 (USAMO 2008/2)
Let ABC be an acute, scalene triangle, and let M, N, and P be the midpoints of $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, respectively. Let the perpendicular bisectors of $\overline{AB}$ and $\overline{AC}$ intersect ray AM in points D and E respectively, and let lines BD and CE intersect in point F, inside of triangle ABC. Prove that points A, N, F, and P all lie on one circle.

Walkthrough.

(a) Consider a phantom point F' as the A-Dumpty point, and further let $BF' \cap AM = D'$.

(b) Exploit the properties of F', and try to get $D'A = D'B$.

(c) Carry on the same by considering E', and lastly observe a bit to get done.

Example 21 (XVII Sharygin Correspondence Round P15, *Anant Mudgal and Navilarekallu Tejaswi*)
Let $APBCQ$ be a cyclic pentagon. A point M inside triangle ABC is such that $\angle MAB = \angle MCA$, $\angle MAC = \angle MBA$ and $\angle PMB = \angle QMC = 90°$. Prove that AM, BP, and CQ concur.

Walkthrough.

(a) Firstly, note that we have M as the A-Dumpty in $\triangle ABC$.

(b) Use the properties of M, to get the configuration, and note there can be two situations (excluding the isosceles one).

(c) Take one, and consider $BQ \cap AM = E_1$ and $CP \cap AM = E_2$, and then angle chase (using the angles you already have!) to get some cyclic quadrilaterals involving those points.

(d) Get $E_1 = E_2$, and finish with radical axes.

Example 22 (AIME 2016 I/15)
Circles ω_1 and ω_2 intersect at points X and Y. Line ℓ is tangent to ω_1 and ω_2 at A and B, respectively, with line AB closer to point X than to Y. Circle ω passes through A and B intersecting ω_1 again at $D \neq A$ and intersecting ω_2 again at $C \neq B$. The three points C, Y, D are collinear, $XC = 67$, $XY = 47$, and $XD = 37$. Find AB^2.

There are many ways to get through, but we will motivate the reader to go around the Dumpty way, for obvious reasons.

Walkthrough.

(a) Note that the intersection of AD and BC (say F) lies on line XY.

(b) Observe $(AXYD)$ and $(BXYC)$ with focus on $\triangle FDY$, to extract another cyclic from the picture.

(c) What's special about quadrilateral $AFBY$?

(d) XC and XD are on either side, with XY in between, so in this scenario of Dumpty's, it's quite motivated to consider the product $XC \cdot XD$.

(e) Try to relate the above one with AB^2, using the things you already have in hand; and get done. (Consider the intersection of AB and FY, then work-out.)

Remark. Note that, we also get X is the Y-HM point in $\triangle YAB$ in the above picture, which is indeed the more motivated one to get at first, but the Dumpty observation proves to more useful here. At the very end of **Charac 6**, we also got something similar - HM & Dumpty in the same picture; this forced relevance is to stress on the fact that there will be instances where both the points pop-up, but we need to deal mindfully and consider the hopefully helpful one, at the moment.

Example 23 (Indian Practice TST 2019/1.2)
Let ABC be a triangle with $\angle A = \angle C = 30°$. Points D, E, F are chosen on the sides AB, BC, CA respectively so that $\angle BFD = \angle BFE = 60°$. Let p and p_1 be the perimeters of the triangles ABC and DEF, respectively. Prove that $p \leq 2p_1$.

Walkthrough.

(a) Find the Dumpty point in the picture. (Note that F is a Dumpty point wrt vertex of some triangle.)

(b) p is commensurable in terms of some side, so find it. (Take the midpoint of AC.)

(c) Apply cosine rule to a suitable side in $\triangle DEF$, and use a specific property of F from (a), to get $DE \geq \sqrt{3}BF$. (AM-GM will be used somewhere!)

(d) For the remaining, use inequalities on sides of the involved triangle, and chase the desired of $p \leq 2p_1$.

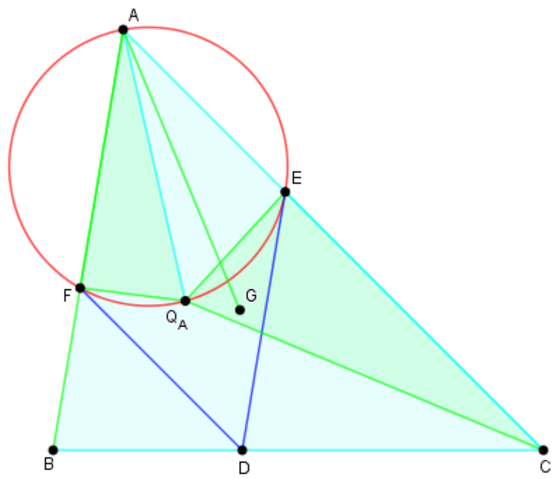
Quite interesting to note, that in most of the examples above, F is the required Dumpty point (and I didn't change point labels). By now, the reader should have realized how important is Equation (1).

4 An Exploration Through Some Nice Problems

Here, we will look into various inter-related problems, with an inclination towards parallel lines and parallelograms. Also, let us not consider a few of the notations we used above, that is, here we will be re-using some of them with different meanings. (Like D, E, F below are not midpoints of respective sides, but points such that $DE \parallel AB$ and $DF \parallel AC$, then, point X in the first part, and so.)

Condition. Let ABC be a triangle with G as its centroid. Let D be a variable point on segment BC. Points E and F lie on sides AC and AB respectively, such that $DE \parallel AB$ and $DF \parallel AC$. Show that,

(I) USA TST 2008/7. As D varies along segment BC, (AEF) passes through a fixed point X such that $\angle BAG = \angle CAX$.



Solution. It's pretty obvious what the starting and only claim could be, and indeed what the point happens to be.

Claim — The required point X, is the A-Dumpty point.

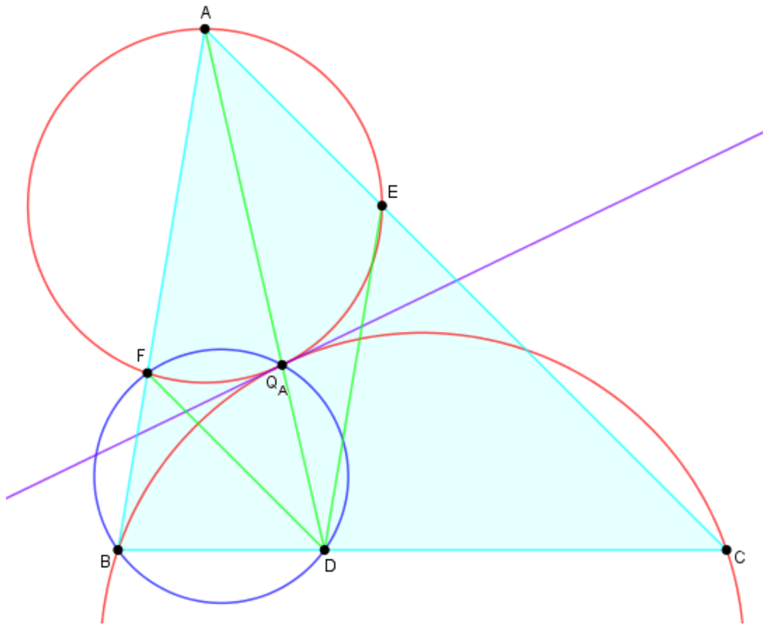
We take X to be defined as in **Charac 4**, and use **Charac 2** to observe that,

$$\frac{BF}{FA} = \frac{BD}{DC} = \frac{AE}{EC},$$

so, the spiral similarity at X takes F to E, that is, $\triangle XFA \sim \triangle XEC$, which implies $\angle XFA = \angle XEC$, and thus, $AFXE$ is cyclic. And as $\angle BAX = \angle GAC$, we're done. □

Let us now re-label X as Q_A.

(II) Winter SDPC 2018-2019 P7 (b). If D lies on line AQ_A, then (AEF) is tangent to (BX_AC).



Solution. So, here we have $D = AQ_A \cap BC$

$$\implies \measuredangle Q_ADF = \measuredangle Q_AAC = \measuredangle Q_ABA = \measuredangle Q_ABF.$$

Hence, we get Q_AFBD as cyclic. (We covered this same circle (Q_ABD) in **Lemma 3**, just there D was labelled as Q.)

Next, we note that

$$\measuredangle FQ_AB = \measuredangle FDB = \measuredangle ACB = \measuredangle ACQ_A + \measuredangle Q_ACB = \measuredangle FAQ_A + \measuredangle Q_ACB,$$

where $\measuredangle$ signifies angles measures modulo 180°. So, appealing to angles in alternate segment, we get that (AEQ_AF) and (BQ_AC) are (externally) tangent to each other at Q_A. $\square$

Sub-condition. If ABC is isosceles with $AB = AC$, and AZ as its circumdiameter.

(III) Latvia TST 2020 Round 1. Then $ZD \perp EF$.

Walkthrough.

(a) Observe that $\triangle BFD$ and $\triangle DEC$ are isosceles, both being directly similar to $\triangle BAC$. (Focus on $\overline{AF}$ and $\overline{AE}$.)

(b) Let the intersections of FD and ED with AZ to be X and Y, respectively. Then, prove that DXY isosceles.

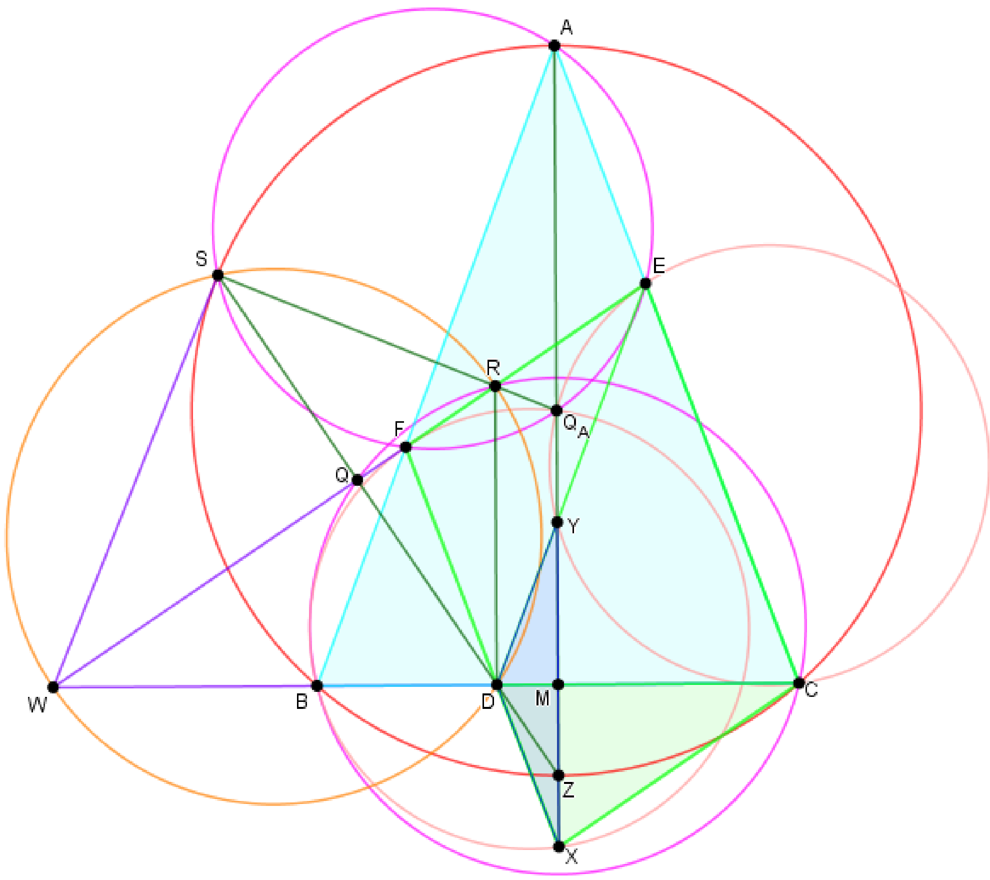
(c) Show that $EC = FX$, and thus, $FXCE$ is parallelogram. (Break CE into sum of other segments.)

(d) Note that Z is the orthocenter of $\triangle DCX$[2], and finish.

[2] This is when X lies outside (ABC), if X lies inside (ABC), then, D would be the orthocenter of $\triangle ZCX$; as it happens in an orthocentric system.

Try the below one.

Lemma 24
Both the quadrilaterals of BFQ_AX and CEQ_AY are cyclic.



Observation. As, $AFEQ_A$ is cyclic, and we also have AQ_A as the bisector of $\angle FAE$, so by Fact 5, we get Q_A as the midpoint of arc EF opposite to A. Note, since the triangle is isosceles, the A-symmedian chord is the AZ itself, with the A-Dumpty point coinciding with the circumcenter.

(IV) Peru EGMO TST 2020/5. If line EF meets DZ at Q and the bisector of $\angle EDF$ at R, then B, Q, R, C are concyclic.

Walkthrough.

(a) Let $AS \parallel EF$, where $S = (ABC) \cap (AEF) \neq A$, and observe that there exists a spiral similarity centered at S (σ) that takes FE to BC (well known). Prove that it takes R to D.[3]

(b) Angle chase to get $\angle CDR = 90^\circ$, and that σ taking RD to Q_AZ is a homothety.

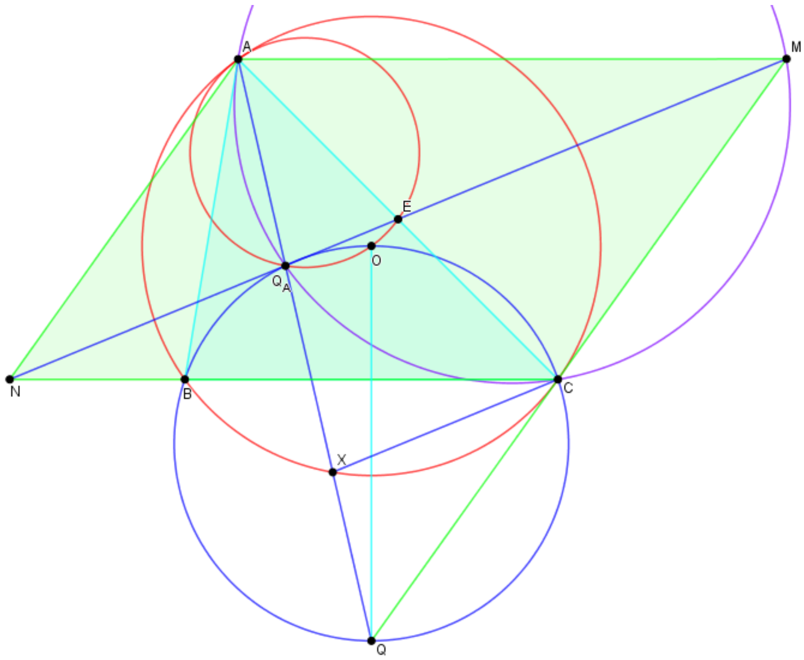
(c) Lastly, let EF meet BC at W, and observe quadrilateral $WSRD$.

[3]Hint: Use angle-bisector theorem & ratios, and the Gliding principle, to note that $\triangle SFB \sim \triangle SRD \sim \triangle SEC$.

The below ones are the last set of related configurations here. So, let's proceed. Readers are advised to first try the just below one.

Problem 12 (Peru TST 2006/4, Dutch IMO TST 2019 2.3). Let ABC be an acute triangle with O as the circumcenter. Point Q lies on (BOC), so that OQ is a diameter, and points M, N lies on CQ, BC respectively, such that $ANCM$ forms a parallelogram. Prove that (BOC), AQ and NM pass through a common point.

Solution is basically the single and obvious claim here, as follows. Here also, let's not consider a



few of the notations we used above, as we will be re-using some of them with different meanings. (Like X and Q.)

Solution.

Claim — Common point is the A-Dumpty point.

We know that Q constitutes the intersection of tangents at B, C to (ABC), and with **Charac 4**, we get $AQ \cap (BOC)$ as the A-Dumpty point. Let's denote it by Q_A (as usual). And further let E to be the midpoint of CA, and $AQ \cap (ABC) = X$. It's just remains to show that $Q_A \in MN$.

Proof.

$$\angle AQ_AC = 180^\circ - \angle CQ_AQ = 180^\circ - \angle CBQ = 180^\circ - \angle CMA$$

So, $M \in (AQ_AC)$. Whence,

$$\angle AQ_AM = \angle ACM = \angle AXC,$$

and thus $Q_AM \parallel XC$. But, $Q_AE \parallel XC$, so Q_A, E, M are collinear. $AMCN$ being a parallelogram, $E \in MN$, and thus we're done. □

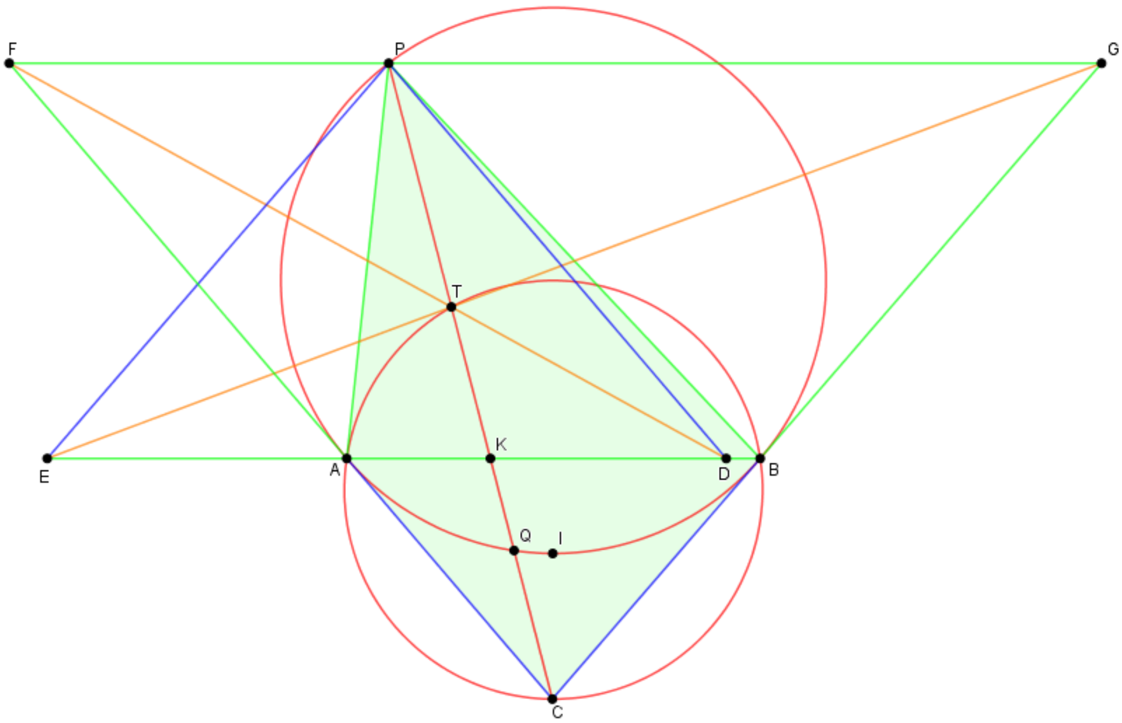
Now, have a look at next one.

Problem 13 (IMO Shortlist 2003 G5, *Hojoo Lee*). Let ABC be an isosceles triangle with $AC = BC$, whose incentre is I. Let P be a point on the circumcircle of the triangle AIB lying inside the triangle ABC. The lines through P parallel to CA and CB meet AB at D and E, respectively. The line through P parallel to AB meets CA and CB at F and G, respectively. Prove that the lines DF and EG intersect on the circumcircle of the triangle ABC.

We will reinterpret the situation in terms of $\triangle PAB$; then, C is the intersection of the tangents at A, B to (PAB).

Comment. P is taken outside $\triangle ABC$ to make the connection more explicit; the proof is the same.

Method I. After the reinterpretation, it's pretty clear from **Problem 12** above, that the intersection point is the P-Dumpty point in $\triangle PAB$.



But, we present another solution built on homothety, which is quite interesting to note.

Method II (by Jeffrey Kwan). Let PC intersect (ABC) at T, AB at K, and (PAB) at Q.

Claim — T is the desired point of intersection.

Using **Lemma 3**, we get

$$CP \cdot CQ = CB^2 = CK \cdot CT \implies \frac{CQ}{CT} = \frac{CK}{CP}. \quad (*)$$

Observe that, there exists a homothety that takes $\triangle PDE$ to $\triangle CFG$, with center at $X = PC \cap DF \cap EG$ (which means concurrency of the three).

We note that

$$\frac{PX}{XC} = \frac{DE}{FG}.$$

Finally,

$$\frac{PT}{TC} = 1 - \frac{CQ}{CT} \overset{(*)}{=} 1 - \frac{CK}{CP} = \frac{KP}{CP} = \frac{DE}{FG}$$

so, $X = T$.

For the last equality, we used $\triangle PDE \cup K \sim \triangle CFG \cup P$ (which basically means that $PDEK$ and $CFGP$ are similar figures). $\square$

Behaviour of Dumpty point under Inversion

Lastly, here is a small note on the aforementioned.

- Inversion wrt (ABC) sends (OQ_ABC) to line BC, and Q_A to K, yielding $\angle OQ_AA = \angle OAK = 90^\circ$.

- When we perform an inversion at vertex A of $\triangle ABC$ with power $r^2 = AB \cdot AC$, it takes B, C, Q_A to B', C', Q'_A, such that $AB'Q'_AC'$ forms a parallelogram. And further, O to the reflection of A over $B'C'$, whence $\angle AO'Q'_A = 90^\circ$, which implies $\angle AQ_AO = 90^\circ$. (Try the same for $\sqrt{\frac{bc}{2}}$, and observe.)

(For the behaviour of HM point under inversion, see here. Readers are suggested to check the discussion here, and a solution here.)

5 More Contest Practice

Problem 14 (CMC 4, CIME II/15). Let ABC be an acute triangle with $AB = 2$ and $AC = 3$. Let O be its circumcenter and let M be the midpoint of $\overline{BC}$. It is given that there exists a point P on (BOC) such that $\angle APB = \angle APC$ and $\angle AOM = \angle APM$. Then $BC^2 = \frac{m}{n}$ for relatively prime positive integers m and n. Find $m + n$.

Problem 15 (STEMS 2020 Category A/12). Let ABC be a triangle with $AB = 4, AC = 9$. Let the external bisector of $\angle A$ meet (ABC) again at $M \neq A$. A circle with center M and radius MB meets the internal bisector of angle A at points P and Q. Determine the length of PQ.

Problem 16 (Arab 2020/2). Let ABC be an oblique triangle and H be the foot of altitude passing through A. Let I, J, K denote the midpoints of segments AB, AC, IJ, respectively. Show that the circle c_1 passing through K and tangent to AB at I, and the circle c_2 passing through K and tangent to AC at J, intersect at second point K', and that H, K and K' are collinear.

Problem 17 (Greece 2018/2). Let ABC be an acute triangle with $AB < AC < BC$ and $c(O, R)$ the circumscribed circle. Let D, E be points in the small arcs AC, AB respectively. Let K be the intersection point of BD, CE and N the second common point of the circumscribed circles of the triangles BKE and CKD. Prove that A, K, N are collinear if and only if K belongs to the symmedian of ABC passing from A.

Problem 18 (INMO 2020/1). Let Γ_1 and Γ_2 be two circles of unequal radii, with centres O_1 and O_2 respectively, intersecting in two distinct points A and B. Assume that the centre of each circle is outside the other circle. The tangent to Γ_1 at B intersects Γ_2 again in C, different from B; the tangent to Γ_2 at B intersects Γ_1 again at D, different from B. The bisectors of $\angle DAB$ and $\angle CAB$ meet Γ_1 and Γ_2 again in X and Y, respectively. Let P and Q be the circumcentres of triangles ACD and XAY, respectively. Prove that PQ is the perpendicular bisector of the line segment O_1O_2.

Problem 19 (Canada 2015/4). Let ABC be an acute triangle with circumcenter O. Let I be a circle with center on the altitude from A in ABC, passing through vertex A and points P and Q on sides AB and AC. Assume that $BP \cdot CQ = AP \cdot AQ$. Prove that I is tangent to (BOC).

Problem 20 (IMO 2014/4). Let P and Q be on segment BC of an acute triangle ABC such that $\angle PAB = \angle BCA$ and $\angle CAQ = \angle ABC$. Let M and N be the points on AP and AQ, respectively, such that P is the midpoint of AM and Q is the midpoint of AN. Prove that the intersection of BM and CN is on the circumference of triangle ABC.

Problem 21 (Iran TST 2015 1.1.2). I_b is the B-excenter of the triangle ABC and ω is the circumcircle of this triangle. M is the middle of arc BC of ω which doesn't contain A. MI_b meets ω at $T \neq M$. Prove that $TB \cdot TC = TI_b^2$.

Problem 22 (IGO Medium 2016/5). Let the circles ω and ω' intersect in points A and B. The tangent to circle ω at A intersects ω' at C and the tangent to circle ω' at A intersects ω at D. Suppose that the internal bisector of $\angle CAD$ intersects ω and ω' at E and F, respectively, and the external bisector of $\angle CAD$ intersects ω and ω' at X and Y, respectively. Prove that the perpendicular bisector of XY is tangent to (BEF).

Problem 23 (IMO Shortlist 2015 G4). Let ABC be an acute triangle and let M be the midpoint of AC. A circle ω passing through B and M meets the sides AB and BC at points P and Q respectively. Let T be the point such that $BPTQ$ is a parallelogram. Suppose that T lies on the circumcircle of ABC. Determine all possible values of $\frac{BT}{BM}$.

Problem 24 (USAJMO 2015/5). Let $ABCD$ be a cyclic quadrilateral. Prove that there exists a point X on segment $\overline{BD}$ such that $\angle BAC = \angle XAD$ and $\angle BCA = \angle XCD$ if and only if there exists a point Y on segment $\overline{AC}$ such that $\angle CBD = \angle YBA$ and $\angle CDB = \angle YDA$.

Problem 25 (IMO Shortlist 2011 G6). Let ABC be a triangle with $AB = AC$ and let D be the midpoint of AC. The angle bisector of $\angle BAC$ intersects the circle through D, B and C at the point E inside the triangle ABC. The line BD intersects the circle through A, E and B in two points B and F. The lines AF and BE meet at a point I, and the lines CI and BD meet at a point K. Show that I is the incentre of triangle KAB.

Problem 26 (IGO 2014/3). A tangent line to circumcircle of acute triangle ABC ($AC > AB$) at A intersects with the extension of BC at P. O is the circumcenter of $\triangle ABC$. Point X lying on OP such that $\measuredangle AXP = 90^\circ$. Points E and F lying on AB and AC, respectively, and they are in one side of line OP such that $\measuredangle EXP = \measuredangle ACX$ and $\measuredangle FXO = \measuredangle ABX$. K, L are points of intersection of EF with (ABC). Prove that OP is tangent to (KLX).

Problem 27 (China TST 2019 2.2.5). Let M be the midpoint of BC of triangle ABC. The circle with diameter BC, ω, meets AB, AC at D, E respectively. P lies inside $\triangle ABC$ such that $\angle PBA = \angle PAC, \angle PCA = \angle PAB$, and $2PM \cdot DE = BC^2$. Point X lies outside ω such that $XM \parallel AP$, and $\frac{XB}{XC} = \frac{AB}{AC}$. Prove that $\angle BXC + \angle BAC = 90^\circ$.

Problem 28 (Iranian TST 2019 2.1.2). In a triangle ABC, $\angle A$ is 60°. On sides AB and AC we make two equilateral triangles (outside the triangle ABC) ABK and ACL. CK and AB intersect at S , AC and BL intersect at R, BL and CK intersect at T. Prove the radical centre of circumcircle of triangles BSK, CLR and BTC is on the median of vertex A in triangle ABC.

Problem 29 (China TST 2021 1.2.5). Given a triangle ABC, a circle Ω is tangent to AB, AC at B, C, respectively. Point D is the midpoint of AC, O is the circumcenter of triangle ABC. A circle Γ passing through A, C intersects the minor arc BC on Ω at P, and intersects AB at Q. It is known that the midpoint R of minor arc PQ satisfies that $CR \perp AB$. Ray PQ intersects line AC at L, M is the midpoint of AL, N is the midpoint of DR, and X is the projection of M onto ON. Prove that the circumcircle of triangle DNX passes through the center of Γ.

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Further Read

For a brief overview on the ***Artzt Parabola*** check out the post on "Parabola with Focus at Dumpty Point, and Directrix Perpendicular to A-median & NPC"; and also here. Is the parabola here same as the Dumpty/Artzt parabola? [4]
The reader might want to explore and have a look here, here, and here.

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India,
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[4]Yes, indeed. It's Exercise 100 of Chapter 2 of **Conic Sections Treated Geometrically** by **W.H. Besant**. Thanks to *David Altizio* for it.