Euler's Sum of Powers Conjecture

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Fermat's Last Theorem, conjectured in 1637, was proved in 1995. It states that the equation $x^n + y^n = z^n$ has no solutions in integers, for $n > 2$, more precisely for $n \geq 3$, other than the trivial solutions in which at least one of the variables is zero.
The **Euler's Sum of Powers Conjecture** attempted to generalize Fermat's last theorem. It states that if the sum of n, kth powers is itself a kth power, then n must be greater than or equal to k. In other words, atleast n, kth powers are required for $n > 2$ to provide a sum that is itself a kth power.
That is, if

$$\sum_{i=1}^{n} a_i^k = b^k,$$

then $n \geq k$. Here, $n > 1, k > 1, a_1, a_2, ..., a_n, b$ are all positive integers.

$$a_1^k + a_2^k + a_3^k + ... + a_n^k = b^k \implies n \geq k.$$

Euler's conjecture holds for $k = 3$, which follows from Fermat's last theorem for the third powers. For example, $3^3 + 4^3 + 5^3 = 6^3$.
Euler's proposition was conjectured in 1769, and it was disproved in 1966 by *L. J. Lander* and *T. R. Parkin* with the counter example $27^5 + 84^5 + 110^5 + 133^5 = 144^5$. In the following years, Lander, Parkin, John Selfridge, Noam Elkies, Roger Frye and many others came up with more such examples that disproved Euler's conjecture for higher values of k. In fact, in 1967 Lander, Parkin and Selfridge came up with their own conjecture, as an extension of Euler's conjecture and Elkies constructed an infinite series of counter examples for the $k = 4$ case.

The example proposed by Lander and Parkin appeared as a problem in *1989,* in the *American Invitational Mathematical Examination.*

The 1989 AIME Problem [P9]:

Euler's conjectures was disproved in the 1960s by three American mathematicians when they showed there was a positive integer such that $27^5 + 84^5 + 110^5 + 133^5 = n^5$. Find the value of n.

Solution:

Clearly, $n > 133$. As, 133 is the largest number in the LHS, so, n must be greater than 133.

As we have got the lower limit, now we must look for the upper limit. So, for that purpose, divide every number on the LHS (without considering the powers), by the smallest number which is 27. On doing so, we get: $\frac{27}{27} = 1$,$\frac{84}{27} = 3.11$, $\frac{110}{27} = 4.07$ and $\frac{133}{27} = 4.92$.

Now, observe that

$27^5 + 84^5 + 110^5 + 133^5 < 27^5(1^5 + 4^5 + 5^5 + 5^5) = 27^5(7275)$

$\implies n^5 < 27^5(7776) \implies n^5 < 27^5 * 6^5 \implies n^5 < (27 * 6)^5$

So, $n < 162$. Therefore, $133 < n < 162$.

Now, let us analyze the LHS of the equation with modulo 2.

So, $27^5 + 84^5 + 110^5 + 133^5 \equiv 1^5 + 0^5 + 0^5 + 1^5 \equiv 2 \equiv 0\ (mod\ 2)$. Thus, n is even. That is, $n \equiv 0\ (mod\ 2)$.

Now, let us do the same with modulo 3.

$27^5 + 84^5 + 110^5 + 133^5 \equiv 0^5 + 0^5 + 2^5 + 1^5 \equiv 33 \equiv 0\ (mod\ 3)$

So, $n \equiv 0\ (mod\ 3)$.

Now, let us analyze with modulo 5.

$27^5 + 84^5 + 110^5 + 133^5 \equiv 2^5 + (-1)^5 + 0^5 + (-2)^5 \equiv -1 \equiv 4\ (mod\ 5)$

So, $n \equiv 4\ (mod\ 5)$.

Now, with all these conditions we can find n in that range. Now, as n is even, let us assume $n = 2k$, for some integer k.

Also, $n \equiv 0 \ (mod\ 3) \implies 2k \equiv 0 \ (mod\ 3)$. On multiplying both sides of the modulo by 2, we get $4k \equiv 0 \ (mod\ 3) \implies k \equiv 0 \ (mod\ 3)$. So, let $k = 3l$, for some integer l.

Now as, $n = 2k$ and $k = 3l$ .Therefore, $n = 6l$. Again as, $n \equiv 4 \ (mod\ 5)$, we have $6l \equiv 4 \ (mod\ 5) \implies l \equiv 4 \ (mod\ 5)$. So, let $l = 5m + 4$, for some integer m.

So, at last, we have $n = 2k = 2(3l) = 6l = 6(5m + 4) = 30m + 24$. Now, let us put the value of n in terms of m in the inequality $133 < n < 162$ $\implies 133 < 30m + 24 < 162$.

On solving, we get $m = 4$. At $m = 3$ or 5, we get 114 and 174 respectively, both of which falls out of the range. So, $m = 4$ is the only possible value. And at $m = 4$, we get $n = 30m + 24 = 144$. $\square$