Sierpinski Triangle

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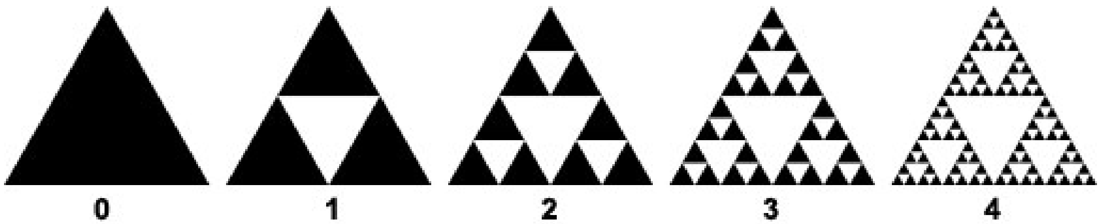
Consider an equilateral triangle with unit side length. Subdivide it into four smaller congruent triangles and remove the central triangle. Repeat the last step with each of remaining smaller triangles.



Now denote the perimeter and area of the existing portion of the triangle at the n-*th* step by $P(n)$ and $A(n)$ respectively. For example, $P(2) = 9/2$ units and $A(2) = 3\sqrt{3}/16$ sq.units.
It can be proved by induction on n that

$$P(n+1) = \frac{3}{2}P(n) \text{ and } A(n+1) = \frac{3}{4}A(n)$$

holds for every $n \geq 1$.
Hence, we get $P(n) = P(1)\left(\frac{3}{2}\right)^{n-1}$ and $A(n) = A(1)\left(\frac{3}{4}\right)^{n-1}$ for every $n \geq 1$.
Now, since $0 < 3/4 < 1$, we know that $(3/4)^n \to 0$ as $n \to \infty$, which implies that $\lim_{n\to\infty} A(n) = 0$. On the other hand, since $3/2 > 1$, we know that $(3/2)^n$ diverges to $+\infty$ as $n \to \infty$. Therefore, we can write $\lim_{n\to\infty} P(n) = \infty$.

Comment: The result might be little surprising, because the limiting figure has an infinite perimeter but its area is zero. This limiting (fractal) figure is known as the Sierpinski Triangle. It is also known as Sierpinski Sieve. You may find more intriguing results here and here.